A PROBLEM IN THE HEATING OF SMALL AREAS ON A LARGE SOLID

by

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Summary

The transient temperature at the centre of a heated circular area on a semi-infinite solid with cooling to the atmosphere has been computed.

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Introduction

In P.R. Note 163/1955 (1) Blok's (2) analysis for the transient heating of a circular area without cooling was given. Steady state temperatures with cooling were also derived. The present note gives the temperatures at the centre of circular area at various times with different degrees of cooling. The work is relevant to the use of small irradiated areas to represent large areas and allows some estimate to be made of the time for which the small area behaves as a large one.

Theory

Let Θ be the temperature
l the incident flux
H the cooling coefficient
K the thermal conductivity
ρ the density
o the specific heat
γ the thermal diffusivity
x, y, z cylindrical coordinates
r the radius

We make use of Green's function for a point source at x', y', z' in a semi-infinite solid (x>0) with radiation at the boundary x = 0 to a medium at zero temperature. This function is given by Carslaw and Jaeger in "The Conduction of heat in solids" (3). If u is the temperature due to an instantaneous point source of unit strength at x', y', z', the temperature at x, y, z after a time t is

\[ u = \frac{1}{8\pi Rt^{3/2}} \left\{ e^{-\frac{(x-x')^2}{4Rt}} + e^{-\frac{(y-y')^2}{4Rt}} - \frac{(y-y')^2 + (z-z')^2}{4Rt} \right\} e^{-\frac{(x-x')^2}{4Rt}} \]

For a continuous source of strength \( \frac{I}{\rho c} \) (over v R) at x' = 0, and \( \sqrt{x'^2 + y'^2} \) equal to \( \gamma \), the temperature at x, y = z = 0 at time t is, from (1).

\[ \Theta = \frac{1}{\rho c} \int_{0}^{t} d\tau \left[ \int_{0}^{2\pi} d\phi \int_{0}^{\infty} dR e^{-\frac{\gamma^2 R^2}{44R\kappa}} \left( \frac{1}{(\pi R\kappa)^\frac{3}{2}} - \frac{\kappa}{\pi R^2} e^{-\frac{\kappa^2 R^2}{44R\kappa}} \right) \right] \]

\[ \int_{0}^{t} d\tau \left( 1 - e^{-\frac{\gamma^2 R^2}{44R\kappa}} \right) \left( \frac{1}{(\pi R\kappa)^\frac{3}{2}} - \frac{\kappa}{\pi R^2} e^{-\frac{\kappa^2 R^2}{44R\kappa}} \right) \]

\[ \frac{I}{\rho c} \int_{0}^{t} d\tau \left( 1 - e^{-\frac{\gamma^2 R^2}{44R\kappa}} \right) \left( \frac{1}{(\pi R\kappa)^\frac{3}{2}} - \frac{\kappa}{\pi R^2} e^{-\frac{\kappa^2 R^2}{44R\kappa}} \right) \]
The solution to the integral of the first term in the first bracket in equation (3) is the result for a heated disk with no surface cooling and is equal to Blok's (2) result

\[ \Theta_{H=0} = \frac{IR}{K} \left\{ \frac{2}{\sqrt{\pi}} \left( \frac{R+1}{R^3} \right)^{1/4} \left( 1 - e^{-R^2/4Kt} \right) + e^{t/2} \frac{R}{K} \right\} \]

Hence

\[ \Theta_{H=0} = \frac{IR}{\rho c} \int_0^t e^{\frac{R^2}{4Kt}} \sqrt{\frac{R+1}{R^3}} \left( 1 - e^{-\frac{R^2}{4Kx}} \right) dx \]

\[ \Theta_{H=0} = \frac{IR}{\rho c} \left[ 1 + \frac{1}{\rho c} \int_0^t e^{\frac{R^2}{4Kx}} \left( 1 - e^{-\frac{R^2}{4Kx}} \right) dx \right] \]

This has been numerically evaluated and in Fig. (1) are shown the resulting values of \( \Theta \) for various values of \( hR \) and \( \frac{R^2}{K} \). For the steady state the value of \( \Theta \) has been obtained analytically (1) as

\[ \Theta = \frac{IR}{K} \left[ 1 + \frac{1}{R^3} \int_0^t e^{\frac{R^2}{4Kx}} \left( H_1(hR) - Y_1(hR) \right) dx \right] \]

Where \( Y_1 \) is Weber's Bessel function of the second kind of the first order and \( H_1 \) is Struve's function of the first order.

It is possible to find the values of \( hR \) at which the temperature at the centre of the circle is, say, 10% and 20% less than it would be for a plane source. These values of \( hR \) are shown in Fig. (2) as a function of \( hR \), while in Fig. (3) \( hR^2t \) is shown as a function of \( hR \) for 10% & 20% error. For any \( h, R, \) and \( k \) it is thus possible to find the maximum duration of heating for which a small source is sensibly similar to a large source.

Acknowledgment

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References

(1) "Some practical limitations to the use of small irradiated areas on solids for the study of thermal damage". Thomas, P. H. and Simms, D. L. Department of Scientific and Industrial Research and Fire Offices' Committee Joint Fire Research Organization. F.R. Note No. 163/1955.


(4) ibid. P. 55.
FIG. 1. TRANSIENT TEMPERATURE AT CENTRE OF HEATED DISK
FIG. 2. VALUES OF $\frac{kt}{R^2}$ AT WHICH TEMPERATURE OF DISK CENTRE IS 10% AND 20% BELOW THAT OF HEATED PLANE
FIG. 3. VALUES OF $h^2kt$ AS FUNCTION OF $hR$ FOR 10% & 20% DIFFERENCE BETWEEN DISK AND PLANE.